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LAMINAR FLOW BETWEEN PARALLEL PLATES WITH INJECTION OF A REACTANT AT HIGH REYNOLDS NUMBER

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NOMENCLATURE

A, B, arbitrary constants of integration;
D, diffusion coefficient;
f, $= v/\theta$, ratio of nondimensional axial velocity to nondimensional temperature;
g, $= Re_e^{1/2} f$, dependent variable in the inner viscous layer;
K, $= 8Q/(\rho_e V_e^2)$, a constant;
l, separation distance between plates;
N, total number of chemical species present;
P, function defined in equation (1);
p, pressure;
Q, quantity defined in equation (1);
Re_e, $= -V_e\rho_e l/\mu_e$, Reynolds number based on injection velocity at $z = 1$;
Re_w, $= V_w\rho_w l/\mu_w$, Reynolds number based on injection velocity at $z = 0$;
r, nondimensional radial coordinate;
T, temperature;
U, function related to radial component of velocity ($= rU$);
u, $= -U/V_e$, nondimensional function related to radial component of velocity;
V, axial component of velocity;
v, $= V/V_e$, nondimensional axial component of velocity;
w, nondimensional chemical source term for heat release;
w_i, nondimensional source term for mass of chemical species i;
x, $= Re_e^{1/2}(z - z_s)$, independent variable in the inner viscous layer;
Y_i, mass fraction of species i;
z, nondimensional axial coordinate.

Greek symbols

θ, $= T/T_e$, nondimensional temperature defined in equation (2);
λ, thermal conductivity;
μ, coefficient of viscosity;
ρ, density.

Subscripts

e, refers to conditions at $z = 1$;
i, identifies species;
o, identifies solutions in inviscid region;
s, refers to conditions at stagnation plane;
w, refers to conditions at $z = 0$.

INTRODUCTION

STEADY flow of an incompressible fluid in a two-dimensional channel with porous walls through which fluid is injected uniformly, has been studied for large Reynolds numbers of injection, the injection velocities being allowed to differ at each wall [1]. Heat transfer in such flows has been considered only for symmetrical configurations in which the injection velocities are the same at each wall [2]. Injection of a species that can rapidly experience an exothermic chemical reaction with the material injected from the opposite wall has not been analyzed under these conditions. The objective herein is to provide such an analysis for nonswirling, axisymmetric flows not constrained by the symmetry condition of equal injection.

The motivation for the work is to aid in interpreting experiments on diffusion flames in counterflow configurations above liquid and solid fuels [3–5]. Such experiments have been analyzed through a boundary-layer approximation with an external velocity gradient obtained from a mass balance developed heuristically. The present study offers a correction to the methods of data reduction that have been employed. Other examples where this analysis can be used include the diffusion flame formed in the internal viscous layer that occurs when a gaseous fuel is injected from one wall and a gaseous oxidant from the other [6], and heterogeneous combustion at the surface of a solid fuel, such as carbon [7], exposed to a counterflow of gaseous oxidant.

FORMULATION

Equations for conservation of mass, momentum, energy and chemical species in axisymmetric coordinates are well known [8]. Coordinates will be nondimensionalized with respect to the separation distance l between the plates, which will be located at $z = 0$ and at $z = 1$. Subscripts w and e identify conditions at $z = 0$ and at $z = 1$, respectively. Reynolds numbers based on injection velocities, $Re_e = -V_e\rho_e l/\mu_e$ and $Re_w = V_w\rho_w l/\mu_w$, appear, V being the velocity in the positive z direction. There exist solutions for which the radial velocity is $rU(z)$, while all other flow quantities are functions only of z, with the exception of the pressure p, which according to the radial component of the equation of momentum conservation can be shown to be given by

$$p = P(z) - r^2 Q(z), \quad (1)$$

a result consistent with the equation of state for an ideal gas only if fractional changes of p with r are negligible in comparison with the absolute pressure (low Mach number). The axial component of momentum conservation implies that Q must be a constant.

Assumptions include neglect of buoyancy, a binary diffusion approximation with a single diffusion coefficient D, Lewis and Prandtl numbers of unity, constant average molecular weights and specific heats, and constancy of the product $\rho\mu$. Quantities are nondimensionalized with respect to their values at $z = 1$, so that

$$\frac{T}{T_e} = \frac{\rho_e}{\rho} = \frac{\mu}{\mu_e} = \frac{\lambda}{\lambda_e} = \frac{\rho D}{\rho_e D_e} \equiv \theta(z), \quad (2)$$

where λ denotes the thermal conductivity and T of course, temperature. Velocities are nondimensionalized with respect to $-V_e$, i.e. $v(z) = V/V_e$ and $u(z) = -U/V_e$. Equations for mass conservation, the radial component of momentum conservation, energy conservation and species conservation become, respectively,

$$u = \theta f'/2, \quad (3)$$

$$\frac{2}{Re_e}[\theta^2 f''' + 3\theta\theta' f'' + (\theta'^2 + \theta\theta'')f'] + 2\theta f f'' + 2\theta' f f' - \theta f'^2 + K = 0, \quad (4)$$

$$\frac{1}{Re_e}(\theta\theta''+\theta'^2)+\theta' f+\frac{w}{Re_e}=0 \tag{5}$$

and

$$\frac{1}{Re_e}(\theta Y_i''+\theta' Y_i')+Y_i' f+\frac{w_i}{Re_e}=0,\ i=1,\ldots,N, \tag{6}$$

where primes denote differentiation with respect to $z, f = v/\theta$, $K = 8Q/\rho_e V_e^2$, w is the chemical source term for heat release, nondimensionalized through multiplication by $l^2/\lambda_e T_e$, w_i denotes the similarly nondimensionalized source term for mass of chemical species i, and Y_i represents the mass fraction of species i. The axial component of momentum conservation provides an equation for $P(z)$, which is not coupled to the rest of the system and which therefore need not be considered further.

Boundary conditions to be applied at $z = 0$ are $f' = 0$ (no slip), $f = \rho_w V_w/\rho_e V_e$, $\theta = T_w/T_e$, $Y_i = Y_{iw}$. Those at $z = 1$ are $f' = 0, f = 1, \theta = 1, Y_i = Y_{ie}$. Since the value of the constant K is to be determined as part of the solution, it is appropriate that these $2N+6$ conditions are available for the system of order $2N+5$, given in equations (4)–(6). Although in realistic problems boundary conditions often are more complex than those given here, involving, for example, interface conservation conditions, such aspects are irrelevant to present considerations.

INVISCID FLOW

Of interest here is the limit of large Re_e, with quantities such as w, w_i and boundary values not exceeding order unity, except possibly at flame sheets in viscous regions, where large w's impose discontinuities in θ''' and Y_i''. Inviscid equations formally result, with $\theta \equiv \theta_o$ = constant and $Y_i \equiv Y_{io}$ = constant being the solutions to equations (5) and (6). Equation (4) becomes $f'^2 - 2ff'' = K/\theta_o$, the general solution to which is $f = Az^2/4 + Bz + (B^2 - K/\theta_o)/A$, where A and B are arbitrary constants. Since the character of this system is the same as that considered previously [1], viscous layers are known to develop which enable boundary conditions to be satisfied. For large positive values of Re_e, a boundary layer cannot occur at $z = 1$ [1], and therefore in an inviscid region that extends to $z = 1$, the values $\theta_o = 1$, $Y_{io} = Y_{ie}$, $A = -K$ and $B = K/2$ must apply to satisfy boundary conditions. The locations of other regions depend on the order of Re_w. If Re_w is large, of order Re_e, then similarly there exists an inviscid region that extends to $z = 0$, having $\theta_o = \theta_w$, $Y_{io} = Y_{iw}$, $A = -K/(\theta_w f_w)$ and $B = 0$. Between these two inviscid regions is an interior viscous layer [1]. On the other hand, if Re_w is small, of order $Re_e^{1/2}$ or less, it will be shown below that the second inviscid region disappears, there being a boundary layer adjacent to $z = 0$. The former configuration may occur, for example, with injection of gaseous fuel and gaseous oxidant through opposite plates. The latter corresponds to injection of gaseous oxidant onto a liquid or solid fuel surface.

Re_w OF ORDER Re_e

The internal viscous layer occurs in a first approximation at a position where the axial velocity in each inviscid region vanishes [1]. From the preceding results, by setting $f = 0$ this produces the two equations $1-(K/4)(1-z_s)^2 = 0$ and $f_w^2\theta_w - (K/4)z_s^2 = 0$ for the measure K of the radial pressure gradient and for the location z_s of the viscous region, the "stagnation plane". The solution to this pair of equations gives

$$z_s = [1+(-V_e/V_w)(T_w/T_e)^{1/2}]^{-1} \tag{7}$$

and

$$Q = [(-V_e)\sqrt{\rho_e}+V_w\sqrt{\rho_w}]^2/2. \tag{8}$$

The structure of the viscous layer is described by introducing the stretched variable $x = Re_e^{1/2}(z-z_s)$. Since axial velocities are small in this layer, the variable $g = Re_e^{1/2}f$ is employed. In these stretched variables, equations (4)–(6) become

$$2[\theta^2 g'''+3\theta\theta' g''+(\theta'^2+\theta\theta'')g'] + 2\theta g g''+2\theta' g g'-\theta g'^2+K=0, \tag{9}$$

$$\theta\theta''+\theta'^2+\theta' g+(w/Re_e)=0 \tag{10}$$

and

$$\theta Y_i''+\theta' Y_i'+Y_i' g+(w_i/Re_e)=0, \tag{11}$$

where primes now denote differentiation with respect to x. Boundary conditions for equations (9)–(11) are obtained through matching to the inviscid solutions. Specifically, $g' \to 2/(1-z_s)$, $\theta \to 1$ and $Y_i \to Y_{ie}$ as $x \to \infty$, and $g' \to (-V_e/V_w)2z_s/(1-z_s)^2$, $\theta \to T_w/T_e$ and $Y_i \to Y_{iw}$ as $x \to -\infty$. Since there exists a translational invariance, the additional condition $g(0) = 0$ may be adopted to render the solution unique.

The simplest version of the viscous-layer problem just defined is the constant-property case, $\theta_w = 1$, $Y_{iw} = Y_{ie}$, without sources, $w = 0$, $w_i = 0$, which has θ and Y_i identically constant and which has previously been analyzed [1]. Problems of heat transfer (Y_i = constant, $w = 0$) and mass transfer (θ = constant, $w_i = 0$) are next in order of complexity. More complicated still are combustion problems for which the w's do not vanish; a constant-density version of such a problem for the counterflow diffusion-flame has been solved [9]. Equations (9) and (10) demonstrate that since w/Re_e arises as the nondimensional source term, the rate of energy release per unit volume is appropriately nondimensionalized through multiplication by $l\mu_e/(-V_e\rho_e T_e\lambda_e)$, in agreement with previously adopted scaling for related problems [3, 10].

R_w OF ORDER $Re_e^{1/2}$ OR LESS

Equation (7) shows that z_s becomes small as V_w becomes small. If it is assumed that T_w/T_e is of order unity and that the thickness of the viscous layer is of order $1/Re_e^{1/2}$, then this thickness becomes of the same order as z_s when Re_w becomes of order $Re_e^{1/2}$. Possibly at these lower Re_w's, the thickness of the viscous layer may be more nearly of order $1/Re_w^{1/2}$, and this becomes of order z_s when Re_w is of order $(Re_e)^{2/3}$. In either case, if Re_w is decreased while Re_e is held constant at a large value, then the inviscid layer adjacent to $z = 0$ disappears before Re_w becomes of order unity. When the inviscid layer no longer is present, it becomes convenient to redefine the stretched variable as $x = Re_e^{1/2}z$. Equations (9)–(11) still are obtained, as are the previously cited matching conditions for $x \to \infty$, but the matching for $x \to -\infty$ is lost, there being, instead, wall boundary conditions applied at $x = 0$. This boundary layer adjacent to $z = 0$ prevails in particular when Re_w is of order $Re_e^{1/2}$ (see below) and also in the limiting case, $Re_w = 0$, in which there is no mass transfer through the boundary at $z = 0$.

As z_s approaches zero, the inviscid solution produces $K = 4$, consistent with $Q = \rho_e V_e^2/2$, obtained from equation (8). The matching conditions simplify to $g' \to 2$, $\theta \to 1$ and $Y_i \to Y_{ie}$ as $x \to \infty$. The boundary conditions at $x = 0$ become $g' = 0$, $g = -\rho_w V_w[l/\rho_e\mu_e(-V_e)]^{1/2}$, $\theta = T_w/T_e$ and $Y_i = Y_{iw}$. It may be noted that when T_w/T_e is of order unity, the value of g at $x = 0$ according to the condition just quoted is of order $Re_w/Re_e^{1/2}$, thus demonstrating applicability of the current formulation for Re_w of order $Re_e^{1/2}$ or less. Blowoff phenomena that occur when $Re_w/Re_e^{1/2}$ exceeds a critical value of order unity, have been considered [11, 12].

A variety of boundary-layer problems, like those mentioned for the internal viscous layer, thus can be defined. One such problem is the diffusion flame in the stagnation-point boundary layer, employing the flame-sheet approximation, as has been analyzed [13]. By suitable transformations of coordinates it can be shown that equations (9)–(11), with the quoted boundary conditions, are exactly equivalent to the problem that was solved [13], if the velocity gradient at the stagnation point in the free stream, du_e/ds, is set equal to $-V_e/l$. This observation provides a correction of a factor of 2 to equation (7) of [3]. It is in fact evident from equation (3), when use is made of the limiting form of the inviscid solution, $\theta = 1$, $f'(0) = 2$, that $u(0) = 1$, which in dimensional terms implies that $du_e/ds = -V_e/l$. The source of the earlier difficulty [3] lay in assigning a uniform axial profile of radial velocity to the inviscid flow. In fact, according to the inviscid solution developed herein, the profile is linear, $ru = r(1-z)$, the inviscid flow being rotational.

EXPERIMENTAL COMPARISONS

Velocity profiles have been measured in two experiments cited previously [6, 7]. Each design had attributes not completely in accord with the systems analyzed herein; the experiment with the internal viscous layer [6] had injection Reynolds numbers in the vicinity of 20, which is sufficiently small for effects of heat conduction on velocities to be measurable, and that with the boundary layer on the burning carbon surface [7] employed an open tube instead of a porous wall, giving a nonzero axial pressure gradient near the injection plane. Approximate corrections for these differences resulted in good agreement between theoretical calculation and measurement [14].

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THE DEVELOPMENT OF A MODEL FOR THREE-DIMENSIONAL FLOW IN TUBE BUNDLES

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NOMENCLATURE

D, tube outside diameter [m];
g_i, component of gravitational acceleration in direction x_i [m/s^2];
k, isotropic flow conductivity [m^2];
K_{ij}, flow conductivity tensor [m^2];
K'_i, flow conductivity in principal direction [m^2];
L, length of bundle measured in flow direction [m];
p, pressure [Pa];
R_{ij}, flow resistance tensor [m^{-2}];
Re, Reynolds number defined by $\rho|\mathbf{u}|D/\mu$;
t, time [s];
u_i, component of superficial velocity in direction x_i [m/s];
U, bundle approach velocity [m/s];
x_i, rectangular coordinate [m];
X, transverse pitch [m];
Y, longitudinal pitch [m].

Greek symbols
α, angle of rotation of axes;
Δp, pressure drop [Pa];
ε, porosity or bundle void fraction;
μ, fluid viscosity [Ns/m^2];
ρ, fluid density [kg/m^3].

1. INTRODUCTION

THE PREDICTION of the velocity and pressure fields for flow outside tubes (or rods) arranged in regular arrays is of considerable practical importance in the design of certain types of heat-transfer equipment. The purpose of this note is to show how pressure drop data for one-dimensional flow in tube bundles may be generalized in order to provide a framework for analyzing the real flow problems which are often multidimensional in nature. The proposed equations are extensions to those previously used for analyzing flow in anisotropic porous media. This approach does not give very fine detail of the flow field, such as the local velocity profile between a pair of tubes, but instead gives the general trends in